Current responses and voltage fluctuations in Josephson-junction systems

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We consider arrays of Josephson junctions as well as single junctions in both the classical and quantum-mechanical regimes, and examine the generalized (frequency-dependent) resistance, which describes the dynamic responses of such Josephson-junction systems to external currents. It is shown that the generalized resistance and the power spectrum of voltage fluctuations are related via the fluctuation-dissipation theorem. Implications of the obtained relations are also discussed in various experimental situations.

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There has been much interest in the dynamics of Josephson junctions [1] and Josephson-junction arrays [2], e.g., current-voltage characteristics, dynamic resistivity, and voltage fluctuations. Among these, the voltage fluctuations provide direct information about the dynamic correlations in equilibrium [3,4], whereas the resistivity probes the response to external currents [5]. The latter is also closely related to the relaxation function, which describes the relaxation behavior towards the equilibrium state. These two probes are therefore complementary to each other, and one may expect, in view of the general idea of the fluctuation-dissipation (FD) theorem, that there exists a FD relation between them. Nevertheless most existing studies have been devoted either to the resistivity or to the voltage fluctuations, and the relation between the two has hardly been investigated. Here we thus make use of the linear response theory to derive the generalized frequency-dependent resistance, and examine the relation between the generalized resistance and the power spectrum of the voltage fluctuations in Josephson-junction systems.

There are three energy scales in a Josephson-junction system: the Josephson coupling energy $E_J \equiv \hbar I_J/2|e|$, the self-charging energy $E_0 \equiv e^2/2C_0$, and the junction-charging energy $E_C \equiv e^2/2C$, where I_J is the Josephson critical current and C_0 and C are the self-capacitance and the junction capacitance, respectively. In case that the Josephson energy dominates ($E_0, E_C \ll E_J$), the system at finite temperatures can be described by classical Langevin-type equations of motion for the phases of the order parameter on superconducting grains. For a single junction, in particular, the equation of motion reads

$$\ddot\phi + \gamma\dot\phi + \sin\phi = I + \zeta(t), \tag{1}$$

where ϕ is the phase difference between the two grains, the time has been rescaled in units of the inverse of the Josephson plasma frequency $\omega_p \equiv \hbar^{-1}\sqrt{8E_C E_J}$, I is the external direct-current bias in units of I_J, and the damping parameter γ is related to the shunt resistance R via $\gamma \equiv \hbar\omega_p/2|e|RI_J$. The noise current $\zeta(t)$ (in units of I_J) is characterized by zero mean and correlation $\langle\, \zeta(t)\zeta(t')\,\rangle = 2\gamma T\delta(t-t')$, where T is the temperature in units of E_J.

We introduce the probability distribution function $P(\phi, v, t)$ and write the Fokker-Planck (FP) equation [6], which corresponds to the Langevin equation (1):

$$\frac{\partial}{\partial t}P(\phi,v,t) = L_{FP}(\phi,v)P(\phi,v,t), \tag{2}$$

where $v \equiv \dot\phi$ is the voltage (in units of $\hbar\omega_p/2|e|$) across the junction, and the FP operator L_{FP} is defined to be

$$L_{FP}(\phi,v) \equiv \left[-\frac{\partial}{\partial\phi}v + \frac{\partial}{\partial v}\left(\gamma v + \sin\phi - I + \gamma T\frac{\partial}{\partial v}\right)\right]. \tag{3}$$

Suppose that the system is disturbed by a time-dependent external current $\delta I(t)$ (in units of I_J), which gives the additional term $-\delta I(t)(\partial/\partial v)$ in the FP operator. The resulting change in the average value of v takes the form

$$\delta\langle v(t)\rangle = \int_{-\infty}^{\infty} dt'\, G(t-t')\,\delta I(t'), \tag{4}$$

where the linear response function $G(t)$ is given by [6]

$$G(t) = -\theta(t)\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv\; v\, \exp(tL_{FP})\,\frac{\partial}{\partial v}P_{\rm eq}(\phi,v) \tag{5}$$

with the equilibrium probability distribution $P_{\rm eq}(\phi,v)$. Note that $G(t)$ describes the voltage response of the system to the external current and can be expressed in terms of the correlation function

$$\begin{aligned} G(t) &= \theta(t)T^{-1}\,\langle\, v(t)[v(0)-\alpha(\phi(0),v(0))]\,\rangle \\ &\equiv \theta(t)T^{-1}\tilde C_v(t), \end{aligned} \tag{6}$$

where

$$\alpha(\phi, v) \equiv \frac{1}{P_{\rm eq}(\phi, v)} \left(v + T\frac{\partial}{\partial v} \right) P_{\rm eq}(\phi, v). \qquad (7)$$

In the frequency space, Eq. (6) takes the simple form

$$2T\,\mathrm{Re}\chi(\omega) = S_v(\omega), \qquad (8)$$

where the generalized resistance $\chi(\omega)$ and the voltage power spectrum $S_v(\omega)$ are defined to be the Fourier transforms of $G(t)$ and $\tilde{C}_v(t)$, respectively. Note also that it is the real part of $\chi(\omega)$ (rather than the imaginary part) which characterizes the dissipation across the junction. Equation (8) thus comprises the FD relation in the resistively and capacitively shunted junction (RCSJ) system, connecting the generalized resistance (i.e., dissipation) with the voltage correlation function describing equilibrium fluctuations.

The above relation is to be compared with the standard FD theorem [7], which is applicable to a Hamiltonian system. Namely, when one can explicitly write $P_{\rm eq} = Z^{-1}e^{-H_{\rm eff}/T}$ with a temperature-independent effective Hamiltonian $H_{\rm eff}$, $\alpha(\phi, v)$ in Eq. (7) simply reduces to a constant, giving the standard FD relation. In the RCSJ system, however, the periodicity in ϕ [8] urges the system to have a non-zero probability current even in the stationary state. As a result, the stationary state cannot be described by an effective Hamiltonian, and in general $\alpha(\phi, v)$ becomes a dynamical variable depending on ϕ and v. Similar features have been pointed out in the system without the $\ddot{\phi}$ term [9], where the FD relation between the correlation and response of ϕ (rather than $\dot{\phi}$) has been considered. Equation (8), in contrast, concerns the voltage $v \equiv \dot{\phi}$, which is a physical quantity. Here it is easy to show that the eigenfunction expansion of $P_{\rm eq}$ [6] leads to the identity $\int dv\, (\partial/\partial v)P_{\rm eq}(\phi, v) = 0$, which implies that the (equilibrium) average of $\alpha(\phi, v)$ is just the average voltage: $\langle\, \alpha(\phi, v)\, \rangle = \langle v \rangle \equiv \bar{v}$. It then follows that at long time scales the correlation function in Eq. (6) reduces to the standard voltage correlation function: $\tilde{C}_v(t) \approx C_v(t) \equiv \langle\, [v(t)-\bar{v}][v(0)-\bar{v}]\, \rangle$, which recovers the standard FD theorem with the voltage power spectrum $S_v(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, C_v(t)$.

On the other hand, in systems with the charging energy dominant over the Josephson coupling energy, quantum fluctuation effects should be taken into account, especially at very low temperatures. Such a macroscopic quantum system can be conveniently described by the quantum phase model:

$$H_0 = 4E_C(n + q)^2 - E_J \cos\phi, \qquad (9)$$

where the number n of excess Cooper pairs and the phase difference ϕ are quantum-mechanically conjugate variables ($[n, \phi] = i$), and q is the external gate charge in units of $2e$. For simplicity, we assume that dissipation due to, e.g., quasiparticle tunneling is negligible in this low-temperature regime, and consider the system without the bias current, where q does not change with time. A disturbing current $\delta I(t)$ applied to the system leads to the perturbation Hamiltonian $H_1 = 8E_C(n + q)\, \delta q(t)$, where $\delta q(t) \equiv -K \int^t dt'\;\, \delta I(t')$ with $K \equiv \sqrt{E_J/8E_C}$. The standard quantum theory of linear response then gives the induced voltage across the system in the form

$$\delta\langle v(\omega)\rangle = \chi_q(\omega)\, \delta q(\omega) = \chi(\omega)\, \delta I(\omega), \qquad (10)$$

where $\chi(\omega) \equiv (iK/\omega)\chi_q(\omega)$ and $\chi_q(\omega)$ is the Fourier transform of the retarded Green's function $G^R(t - t') \equiv i\theta(t - t')\, \langle\, [v(t), v(t')]\, \rangle$. The corresponding FD theorem reads

$$(1 - e^{-\omega/KT})S_v(\omega) = 2\, \mathrm{Im}\chi_q(\omega) = 2(\omega/K)\, \mathrm{Re}\chi(\omega), \qquad (11)$$

where the power spectrum $S_v(\omega)$ is again given by the Fourier transform of the voltage correlation function $C_v(t)$, and ω should be understood as $\lim_{\epsilon\to 0^+}(\omega + i\epsilon)$. It is pleasing that in the classical limit ($\omega/T \to 0$), Eq. (11) reproduces the classical relation Eq. (8). Here without the bias current, we have $\bar{v} = 0$ and $C_v(t) = \langle\, v(t)v(0)\, \rangle$. In the presence of the bias current, n_0 increases with time and the unperturbed Hamiltonian H_0 depends explicitly on time, which in general does not allow the standard derivation. Nevertheless when the bias current is sufficiently small, it may be incorporated into the disturbing current; this leads to the same FD relation as that shown in Eq. (11).

We now investigate the physical implications of the relations in Eqs. (8) and (11) to several cases. First, we consider the case that the external bias current is smaller than the Josephson critical current ($I < 1$). In view of Eq. (8), of particular interest in this case is the underdamped ($\gamma \ll 1$) classical junctions at low temperatures. At zero temperature the phase of the (unperturbed) junction stays at one of the local minima $\phi = \sin^{-1} I \pmod{2\pi}$ of the wash-board potential $U(\phi) = -\cos\phi - I\phi$. Small perturbations then induce the well-known plasma oscillation in the vicinity of the local minimum, with the oscillation frequency ω_P given by $(1 - I^2)^{1/4}$ at $T = \gamma = 0$. At finite but sufficiently low temperatures ($T \ll 1$), the noise current stirs up the small fluctuations around the local minimum, and the power spectrum of the voltage is known to become $S_v(\omega) = 2\gamma T \omega^2[(\omega^2 - \omega_P^2)^2 + \gamma^2\omega^2]^{-1}$ [1]. The FD relation in Eq. (8) then gives

$$\mathrm{Re}\chi(\omega) = \frac{\gamma\omega^2}{(\omega^2 - \omega_P^2)^2 + \gamma^2\omega^2}, \qquad (12)$$

which reveals that at low temperatures the generalized resistance is temperature-independent. Note also the behavior $\mathrm{Re}\chi(\omega) \to 0$ as $\omega \to 0$, which is nothing but the manifestation of the superconducting channel. A system with strong quantum fluctuations displays another interesting phenomenon, the Bloch oscillation [10], at sufficiently low temperature. In this case, Eq. (11) indicates

that like the voltage spectrum, the generalized resistance should also exhibit resonance peaks at the Bloch oscillation frequencies proportional to the bias current.

We next consider the Josephson oscillation in an overdamped ($\gamma \gg 1$) Josephson junction with a large bias current ($I > 1$) [11], for which the Fokker-Planck equation (2) reduces to [6]

$$\gamma \frac{\partial}{\partial t} P(\phi, t) = \frac{\partial}{\partial \phi} \left(\sin\phi - I + T \frac{\partial}{\partial \phi} \right) P(\phi, t). \qquad (13)$$

At zero temperature, the voltage $v(t)$ across such a resistively shunted junction (RSJ) displays the Josephson oscillation with the frequency $\omega_J = \gamma^{-1}\sqrt{I^2 - 1}$, which leads to the dc voltage $\bar{v} = \omega_J$. As the temperature is increased from zero, the dc voltage also grows, approaching the Ohmic characteristics. At the same time, thermal fluctuations introduce decaying behavior of the probability in addition to the oscillatory behavior. The eigenfunction expansion of the transition probability [6] at large time scales shows that the leading contribution comes from the lowest eigenvalue, giving

$$P(\phi, t|\phi_0, 0) \sim e^{-\gamma_J |t|} \cos(\omega_J t) + \text{higher harmonics}, \qquad (14)$$

where the line width γ_J of the Josephson oscillation are related to the dc voltage $\bar{v}(I, T)$ via $\gamma_J = \pi\gamma(d\bar{v}/dI)^2 T$ [1]. While at $T = 0$, we obviously have $\gamma_J = 0$ and $\omega_J = \gamma^{-1}\sqrt{I^2 - 1}$, at finite temperatures, they may be estimated numerically to give $\gamma_J \approx \bar{v} = \omega_J$ at $T \approx 1$ (see also Sec. 11.3 of Ref. [6]). Equation (14) leads to the power spectrum

$$S_v(\omega) \sim \frac{\gamma_J}{\gamma_J^2 + (\omega - \omega_J)^2} + \frac{\gamma_J}{\gamma_J^2 + (\omega + \omega_J)^2} + \text{higher harmonics}, \qquad (15)$$

which shows correlation peaks at $\omega = \pm\omega_J$; this is followed by resonances of the resistance via Eq. (8). It is also of interest to note the crossover behavior to the high-temperature (noise-dominant) regime at $T \approx 1$, where the correlation peak in $S_v(\omega)$ disappears.

Heretofore we have concentrated on single-junction systems, but the generalization to an array system is straightforward. The classical equation of motion (1) is easily generalized to any array of Josephson junctions, e.g., a two-dimensional (2D) square $N \times N$ array:

$$\sum_j C_{ij}\ddot{\phi}_j + \gamma \sum_j \Delta_{ij}\dot{\phi}_j + {\sum_j}' \sin(\phi_i - \phi_j) = I_i + {\sum_j}' \zeta_{ij}(t) \qquad (16)$$

with the noise current characterized by $\langle\, \zeta_{ij}(t)\zeta_{kl}(0)\,\rangle = 2\gamma T\delta(t)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$, where Δ_{ij} is the lattice Laplacian, $C_{ij} \equiv (C_0/C)\delta_{ij} + \Delta_{ij}$ is the dimensionless capacitance matrix, and the prime restricts the summation over the nearest neighbors of site i. I_i is the current fed into site $i = (x, y)$, given by $I_i = I(\delta_{x,0} - \delta_{x,N})$; this corresponds to that along one edge of the array ($x = 0$), uniform current I is injected into each site, while along the opposite edge ($x = N$) the same current is extracted from each site. Similarly, in the quantum regime, the array is described by the Hamiltonian:

$$H_0 = 4E_C \sum_{ij} (n_i + q_i) C_{ij}^{-1} (n_j + q_j) - E_J \sum_{<ij>} \cos(\phi_i - \phi_j) \qquad (17)$$

with $[n_i, \phi_j] = i\delta_{ij}$, which is the obvious generalization of Eq. (9). With these, we can use the same procedure as that for a single junction, which leads to the conclusion that the FD relations in Eqs. (8) and (11) are applicable to junction arrays as well [12].

To examine the implications of the FD relations to array systems, we consider a 2D array at zero (direct) current bias, which is well known to display the Berezinskii-Kosterlitz-Thouless (BKT) transition [13] at $T = T_{BKT}$. According to the dynamic theory of the BKT transition [5,14] and ac electrical measurements [15], the contribution of the vortex bound pairs is screened out by the free vortices and the imaginary (inductive) part of the frequency-dependent complex impedance sharply decreases to zero at frequency-dependent temperature T_ω ($> T_{BKT}$), which is accompanied by a peak of the real (resistive) part. The FD relation in Eq. (8) then suggests that the voltage power spectrum $S_v(\omega)$ as a function of the temperature should also show a peak at $T = T_\omega$. The voltage power spectrum, which, to our knowledge, has not been reported on a 2D array near the BKT transition temperature, can be measured in equilibrium and complement another equilibrium measurement obtaining the flux noise spectrum [16]. In the latter, an interesting relation between the flux noise spectrum and the frequency-dependent conductivity has been examined at $T \approx T_{BKT}$ [17]. It would thus be an interesting topic in the future work to investigate in this region the connection between the voltage spectrum and the flux noise spectrum.

For large bias currents ($I > 1$), the voltage spectrum $S_v(\omega)$ has been studied numerically in 2D RSJ arrays [4], which indeed has revealed the correlation peaks as in Eq. (15). Further, it has been observed that the peaks in $S_v(\omega)$ disappear at temperature $T \approx 1\,(\equiv E_J/k_B)$, which happens to correspond to the BKT vortex unbinding transition temperature in the absence of the bias current. Here we point out that the apparent disappearance of the peak has little to do with the phase transition; rather Eq. (15) shows that it is just a crossover. This is consistent with the result that there should be no phase transition in the 2D array driven by external currents larger than the junction critical current [18].

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